If $z=y$ and $x=a-2y$, $a\le 2\le 3$ we get

$$x^4+y^4+z^4+(xyz)^2-3\left(\frac{a}{3}\right)^4-\left(\frac{a}{3}\right)^6=\frac{(3y-a)^2}{729}P(y,a)$$
$$=\frac{(3y-a)^2}{729}(324y^4-108ay^3+1458y^2-1620ay+702a^2-27a^2y^2-6a^3y-a^4)$$

Now we prove that $P(y,a)>0$. Indeed

$$702a^2-a^4=a^2(702-a^2)\ge a^2(702-9)=693a^2,\qquad 6a^3y=6a^2ay\le 54ay$$
$$1011y^2+693a^2>1674ay=(1620+54)ay$$

so we get

$$324y^4+442y^2\ge 108ay^3+27a^2y^2$$

and this is implied by

$$324y^4+442y^2\ge 108\cdot 3y^3+27\cdot 9y^2 \iff 324y^4+199y^2\ge 324y^3$$

and this finally follows by the AGM $324y^4+199y^2\ge 507y^3$. We have showed that

$$x^4+y^4+z^4+(xyz)^2-3\left(\frac{a}{3}\right)^4-\left(\frac{a}{3}\right)^6\ge 0$$

and the difference equals zero if $x=y=z=a/3$. On account of their definition, x,y,z can be equal if and only if $A=B=C=\pi/6$ whence $x=y=z=1$ and

$$x^4+y^4+z^4+(xyz)^2-4\ge 0$$

The searched minimum is thus $\min\{8,4\}=4$.

Also solved by José Gibergans-Báguena, BARCELONA TECH, Barcelona, Spain.

59. *Let x,y,z be nonzero complex numbers. Prove that*

$$81\left(\frac{1}{|x|^2}+\frac{1}{|y|^2}+\frac{1}{|z|^2}\right)^{-1}\le 3|x+y+z|^2+|2x-y-z|^2+|2y-x-z|^2+|2z-x-y|^2$$

(Training Catalonian Team for OME 2014)

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. More generally, consider n nonzero complex numbers $z_1,\ldots,z_n$, and let $\mu=\frac{1}{n}(z_1+\ldots+z_n)$. Clearly we have

$$\sum_{k=1}^{n}|z_k-\mu|^2=\sum_{k=1}^{n}|z_k|^2-2\sum_{k=1}^{n}\Re(z_k\overline{\mu})+\sum_{k=1}^{n}|\mu|^2$$
$$=\sum_{k=1}^{n}|z_k|^2-2n|\mu|^2+n|\mu|^2=\sum_{k=1}^{n}|z_k|^2-n|\mu|^2$$

That is

$$\sum_{k=1}^{n}|z_k|^2=n|\mu|^2+\sum_{k=1}^{n}|z_k-\mu|^2 \tag{1}$$

Now, the HM-AM inequality shows that

$$n^2\left(\sum_{k=1}^{n}\frac{1}{|z_k|^2}\right)^{-1}\leq\sum_{k=1}^{n}|z_k|^2 \tag{2}$$

Combining (1), (2), and rearranging, we obtain

$$n^4\left(\sum_{k=1}^{n}\frac{1}{|z_k|^2}\right)^{-1}\leq n\left|\sum_{k=1}^{n}z_k\right|^2+\sum_{k=1}^{n}\left|(n-1)z_k-\sum_{j\neq k}z_j\right|^2.$$

The proposed inequality corresponds to $n=3$ and $(z_1,z_2,z_3)=(x,y,z)$.

Solution 2 by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain. We have

$$\left|\frac{x+y+z}{3}\right|^2+\frac{1}{3}\left(\left|\frac{2x-y-z}{3}\right|^2+\left|\frac{2y-x-z}{3}\right|^2+\left|\frac{2z-x-y}{3}\right|^2\right)$$

$$=\frac{(x+y+z)(\overline{x+y+z})}{9}$$

$$+\frac{1}{3}\left[\frac{(2x-y-z)(\overline{2x-y-z})}{9}+\frac{(2y-x-z)(\overline{2y-x-z})}{9}+\frac{(2z-x-y)(\overline{2z-x-y})}{9}\right]$$

$$=\frac{1}{3}\left(|x|^2+|y|^2+|z|^2\right)$$

Applying AM-HM inequality, yields

$$\frac{1}{3}\left(|x|^2+|y|^2+|z|^2\right)\geq\frac{3}{\dfrac{1}{|x|^2}+\dfrac{1}{|y|^2}+\dfrac{1}{|z|^2}}$$

and taking into account that the given inequality is equivalent to

$$\frac{3}{\dfrac{1}{|x|^2}+\dfrac{1}{|y|^2}+\dfrac{1}{|z|^2}}\leq\frac{1}{9}(|x+y+z|^2)$$

$$+\frac{1}{27}\left(|2x-y-z|^2+|2y-x-z|^2+|2z-x-y|^2\right)$$

from which the statement follows. Notice that equality holds when $|x|=|y|=|z|$ and we are done.

Solution 3 by Arkady Alt, San Jose, California, USA. Since

$$\begin{aligned}
|x+y+z|^2&=(x+y+z)\left(\overline{x}+\overline{y}+\overline{z}\right)\\
&=|x|^2+|y|^2+|z|^2+(x\overline{y}+\overline{x}y)+(y\overline{z}+\overline{y}z)+(z\overline{x}+\overline{z}x),\\
|2x-y-z|^2&=(2x-y-z)\left(2\overline{x}-\overline{y}-\overline{z}\right)\\
&=4|x|^2+|y|^2+|z|^2-2\left(x\overline{y}+\overline{x}y\right)+(y\overline{z}+\overline{y}z)-(z\overline{x}+\overline{z}x),\\
|2y-x-z|^2&=(2y-z-x)\left(2\overline{y}-\overline{z}-\overline{x}\right)\\
&=|x|^2+4|y|^2+|z|^2-2\left(y\overline{z}+\overline{y}z\right)+(z\overline{x}+\overline{z}x)-(x\overline{y}+\overline{x}y),\\
|2z-x-y|^2&=(2z-x-y)\left(2\overline{z}-\overline{x}-\overline{y}\right)\\
&=|x|^2+|y|^2+4|z|^2-2\left(z\overline{x}+\overline{z}x\right)+(x\overline{y}+\overline{x}y)-(z\overline{x}+\overline{z}x),
\end{aligned}$$

then

$$3|x+y+z|^2+|2x-y-z|^2+|2y-z-x|^2+|2z-x-y|^2$$
$$=3\left(|x|^2+|y|^2+|z|^2\right)+6\left(|x|^2+|y|^2+|z|^2\right)=9\left(|x|^2+|y|^2+|z|^2\right)$$

and the original inequality becomes

$$81\left(\frac{1}{|x|^2}+\frac{1}{|y|^2}+\frac{1}{|z|^2}\right)^{-1}\le 9\left(|x|^2+|y|^2+|z|^2\right)$$

or

$$9\left(\frac{1}{|x|^2}+\frac{1}{|y|^2}+\frac{1}{|z|^2}\right)^{-1}\le\left(|x|^2+|y|^2+|z|^2\right),$$

where latter inequality holds because by Cauchy's inequality

$$\left(|x|^2+|y|^2+|z|^2\right)\left(\frac{1}{|x|^2}+\frac{1}{|y|^2}+\frac{1}{|z|^2}\right)\ge 9$$

Also solved by José Gibergans-Báguena, BARCELONA TECH, Barcelona, Spain.

60. *Compute the following sum:*

$$\sum_{1\le i_1\le n+1}\frac{1}{i_1}+\sum_{1\le i_1<i_2\le n+1}\frac{1}{i_1 i_2}+\ldots+\sum_{1\le i_1<\ldots<i_n\le n+1}\frac{1}{i_1 i_2\ldots i_n}$$

(Training Spanish Team for VJIMC 2014)

Solution 1 by José Gibergans-Báguena, BARCELONA TECH, Barcelona, Spain. Let us denote by S the following sum:

$$S=1+\sum_{1\le i_1\le n+1}\frac{1}{i_1}+\sum_{1\le i_1<i_2\le n+1}\frac{1}{i_1 i_2}+\ldots+\frac{1}{1\cdot 2\cdots n+1}$$
$$=\prod_{k=1}^{n+1}\left(1+\frac{1}{k}\right)=\prod_{k=1}^{n+1}\frac{k+1}{k}=\frac{(n+2)!}{(n+1)!}=n+2$$

Then

$$\sum_{1\le i_1\le n+1}\frac{1}{i_1}+\sum_{1\le i_1<i_2\le n+1}\frac{1}{i_1 i_2}+\ldots+\sum_{1\le i_1<\ldots<i_n\le n+1}\frac{1}{i_1 i_2\ldots i_n}$$
$$=S-\left(1+\frac{1}{1\cdot 2\cdots n+1}\right)=n+1-\frac{1}{(n+1)!}$$

and we are done. □

Solution 2 by Arkady Alt, San Jose, California, USA. Consider the polynomial

$$P(x)=\prod_{k=1}^{n+1}\left(x+\frac{1}{k}\right)=x^{n+1}+a_{n+1}+\sum_{k=1}^{n}a_k x^{n+1-k}$$